

# PINCHING ESTIMATES FOR SOLUTIONS OF THE LINEARIZED RICCI FLOW SYSTEM IN HIGHER DIMENSIONS

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ABSTRACT. We prove pinching estimates for solutions of the linearized Ricci flow system on a closed manifold of dimension  $n \geq 4$  with positive scalar curvature and vanishing Weyl tensor. If the vanishing Weyl tensor condition is removed, we only give a rough pinching estimate controlled by some blow-up function. These results generalize the 3-dimensional case due to G. Anderson and B. Chow (Calc. Var., 23 (2005), 1-12).

## 1. INTRODUCTION AND MAIN RESULTS

Given an  $n$ -dimensional closed Riemannian manifold  $M^n$ , a smooth family of Riemannian metrics  $g(t)$ ,  $t \in [0, T)$ , is said to be evolving under the *Ricci flow* if

$$(1) \quad \frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

where  $R_{ij}$  is the Ricci curvature of the metric  $g(t)$ . This geometric flow was introduced by R. Hamilton [14]. In [14], he proved that for any an initial metric  $g_0$ , there is a unique solution to this flow over some short time interval, with  $g(0) = g_0$ . This proof was later simplified by D. DeTurck [11] by linearizing the modified Ricci flow, which leads to the following *Lichnerowicz Laplacian heat equation*

$$(2) \quad \frac{\partial}{\partial t} h_{ij} = (\Delta_L h)_{ij} := \Delta h_{ij} + 2R_{ikjl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki},$$

for a symmetric 2-tensor  $h$ , where  $R_{ikjl}$  denotes the Riemannian curvature of the metric  $g(t)$  moving under the Ricci flow. Nowadays the Ricci flow (1) coupled with equation (2) is often called the *linearized Ricci flow system*. If we let  $h_{ij} = R_{ij}$ , then (2) is exactly the evolution of the Ricci curvature under the Ricci flow.

On the subject of differential Harnack inequalities for the linearized Ricci flow system, there have been many important contributions. In [8], B. Chow and R. Hamilton proved a linear trace differential Harnack inequality for the coupled system (1)-(2) by adapting similar techniques of Hamilton's trace inequality for the Ricci flow [15] and Li-Yau's seminal inequality for the heat equation [19]. Meanwhile in [7], B. Chow and S.-C. Chu gave a geometric interpretation in terms of this linear trace differential Harnack inequality. In [12], C. Guenther, J. Isenberg

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and D. Knopf studied the linearized Ricci flow system at flat solutions from the point of view of maximal regularity theory. Besides, the related Li-Yau-Hamilton type differential Harnack inequalities for the linearized Ricci flow system were also appeared in [6], [9] and [22].

In another direction, in [1], G. Anderson and B. Chow considered the quantity  $|h|^2/R^2$  for solutions to the linearized Ricci flow system (1)-(2) on closed 3-manifolds with positive scalar curvature. They proved a pinching estimate for solutions of the linearized Ricci flow system on a closed 3-manifold:

**Theorem A** (Anderson and Chow [1]). *Let  $(M^3, g(t))$  be a solution to the Ricci flow on a closed 3-manifold on a time interval  $[0, T)$  with  $T < \infty$  and let  $\rho \in [0, \infty)$  be such that  $R_{\min}(0) > -\rho$ . If the pair  $(g, h)$  is any solution to the linearized Ricci flow system (1)-(2), then there exists a constant  $C < \infty$  depending only on  $g(0)$ ,  $h(0)$ ,  $\rho$  and  $T$  such that*

$$\frac{|h|}{R + \rho} \leq C$$

on  $M \times [0, T)$ . Furthermore, when  $\rho = 0$ ,  $C$  is independent of  $T$ .

The inspiration for the above pinching estimate partly comes from the works of R. Hamilton (see Section 10 of [14] and Section 24 of [16]) and M. Gursky [13]. We refer the interested reader to the introduction of [1] for nice explanations on this subject.

In Theorem A, if we take  $h = Rc$  and  $\rho = 0$ , we immediately obtain Hamilton's Ricci pinching estimate:

**Theorem B** (Hamilton [14]). *If  $(M^3, g(t))$ ,  $t \in [0, T)$ ,  $T < \infty$  is a solution to the Ricci flow (1) on a closed 3-manifold with positive scalar curvature, then there exists a constant  $C < \infty$  depending only on  $g(0)$  such that*

$$\frac{|Rc|}{R} \leq C$$

on  $M \times [0, T)$ .

Recently, S. Brendle [2] has applied Theorem A (see Proposition 6.1 in [2]) to give a complete proof of the uniqueness of the 3-dimensional Bryant soliton which is  $k$ -noncollapsed and non-flat. This resolves a well-known question mentioned in Perelman's first paper [20]. Later in [3], Brendle generalized his result to the higher dimensional setting. That is, he obtained a similar uniqueness theorem for the steady gradient Ricci soliton of dimension  $n \geq 4$  which has positive sectional curvature and is asymptotically cylindrical. In the proof of the higher dimensions, Brendle adapted the arguments in [2] but needed some extra work. For example, he employed a different argument to prove an estimate related the Lichnerowicz-type equation (see Proposition 4.2 in [3]), which is quite different from the 3-dimensional case. Because so far there is not an analogue of Theorem A in higher dimensions.

In this paper, we mainly generalize Anderson-Chow's pinching estimate to the higher dimensions under some curvature assumptions. Our results could be viewed as a partial answer to a question implied in Brendle's paper [3], which may be expected to study the singularities of the Ricci flow.

On one hand, we will establish an Anderson-Chow's pinching estimate for solutions of the linearized Ricci flow system on a closed manifold of dimension  $n \geq 4$  as long as scalar curvature preserves positive and the Weyl tensor remains identically

zero under the linearized Ricci flow system. The Weyl tensor is an important geometric quantity in understanding higher dimensional Riemannian manifolds. This tensor is known to depend on the conformal structure, so that if  $\tilde{g}_{ij} = \phi g_{ij}$ , then  $\tilde{W}_{ijkl} = \phi W_{ijkl}$ . It is well-known that if  $n \leq 3$ , the Weyl tensor vanishes; if  $n \geq 4$ , in local coordinate, the Weyl tensor can be written as

$$(3) \quad \begin{aligned} W_{ikjl} = & R_{ikjl} - \frac{1}{n-2}(g_{ij}R_{kl} + g_{kl}R_{ij} - g_{il}R_{jk} - g_{jk}R_{il}) \\ & + \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{jk}). \end{aligned}$$

Our first main result gives a precise pointwise measure of the size of  $h$  relative to the scalar curvature in higher dimensions.

**Theorem 1.1.** *Let  $(M^n, g(t), h(t))$ ,  $t \in [0, T)$ ,  $T < \infty$  be a solution to the linearized Ricci flow system (1)-(2) on a closed  $n$ -dimensional ( $n \geq 4$ ) manifold with positive scalar curvature. Assume that the Weyl tensor remains identically zero under the linearized Ricci flow system. Then there exists a constant  $c_0 < \infty$  depending only on  $g(0)$  and  $h(0)$  such that*

$$\frac{|h|}{R}(x, t) \leq c_0$$

on  $M \times [0, T)$ .

The proof of Theorem 1.1 essentially follows the arguments of [1]. The main difficulty is that we need to deal with more complicated curvature terms due to the higher dimensions.

**Remark 1.2.** *If we let  $h = Rc$ , then Theorem 1.1 is just as a special case of Knopf's Ricci curvature pinching estimate [18] (see also [4]).*

**Remark 1.3.** *Recently, X.-D. Cao and H. Tran [5] observed that the Ricci flow solution with nonnegative isotropic curvature implies a Riemannian curvature pinching result:*

$$\frac{|Rm|}{R}(x, t) \leq c$$

for some constant  $c = c(n, g(0)) < \infty$ .

On the other hand, if the vanishing Weyl tensor condition is removed in Theorem 1.1, we will give another Anderson-Chow's type pinching estimate for solutions of the linearized Ricci flow system on a closed manifold. Though this pinching estimate is not uniformed by a constant, it could be controlled by some blow-up function.

**Theorem 1.4.** *Let  $(M^n, g(t), h(t))$ ,  $t \in [0, T)$ ,  $T < \infty$  be a solution to the linearized Ricci flow system (1)-(2) on a closed  $n$ -dimensional ( $n \geq 4$ ) manifold with positive scalar curvature, and let  $K := \max_{t=0} |Rm|$ . Then there exist finite constants  $c_0 := c_0(g(0), h(0))$  and  $c := c(n)$  such that*

$$\frac{|h|^2}{R^2}(t) \leq c_0 \cdot (1 - 8Kt)^{-c}$$

on  $M \times [0, T')$ , where  $T' := \min\{T, \frac{1}{8K}\}$ .

The rest of this paper organized as follows. In Section 2, following the arguments of [1], we will prove Theorem 1.1 by the straightforward computation and the usage of parabolic maximum principle. The main difference is that we need to

estimate the nonnegativity of some complicated terms in the evolution equation due to the higher dimensions. In Section 3, we first give an estimate on the norm of Riemannian curvature under the Ricci flow on a closed manifold. Then we apply this estimate and parabolic maximum principles to prove Theorem 1.4. In Section 4, we will use the basic matrix theory to justify the nonnegativity of the above mentioned terms appeared in the proof of Theorem 1.1.

## 2. PROOF OF THEOREM 1.1

Along the Ricci flow (1), we have

$$(4) \quad \frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2.$$

Combining equations (1), (2), (4), and the key estimate (see Claim 2.1), we complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* The proof involves a direct computation and the parabolic maximum principle. Here we can borrow Anderson-Chow's computation to simplify our calculation. From the evolution equation (16) in [1], we have

$$(5) \quad \begin{aligned} \frac{\partial}{\partial t} \left( \frac{|h|^2}{R^2} \right) &= \Delta \left( \frac{|h|^2}{R^2} \right) + \frac{2}{R} \nabla R \cdot \nabla \left( \frac{|h|^2}{R^2} \right) - \frac{2}{R^4} |R \nabla_i h_{jk} - \nabla_i R h_{jk}|^2 \\ &\quad + \frac{4}{R^2} R_{ikjl} h_{ij} h_{kl} - \frac{4}{R^3} |h|^2 |Rc|^2. \end{aligned}$$

The above formula holds for all dimensions. Since here  $n \geq 4$  and the Weyl tensor  $W = 0$  by assumption, from (3) we have

$$\begin{aligned} R_{ikjl} &= \frac{1}{n-2} (g_{ij} R_{kl} + g_{kl} R_{ij} - g_{il} R_{jk} - g_{jk} R_{il}) \\ &\quad - \frac{R}{(n-1)(n-2)} (g_{ij} g_{kl} - g_{il} g_{jk}). \end{aligned}$$

Then substituting this into (5) yields

$$(6) \quad \begin{aligned} \frac{\partial}{\partial t} \left( \frac{|h|^2}{R^2} \right) &= \Delta \left( \frac{|h|^2}{R^2} \right) + \frac{2}{R} \nabla R \cdot \nabla \left( \frac{|h|^2}{R^2} \right) \\ &\quad - \frac{2}{R^4} |R \nabla_i h_{jk} - \nabla_i R h_{jk}|^2 - \frac{4}{R^3} P, \end{aligned}$$

where

$$\begin{aligned} P &:= |h|^2 |Rc|^2 - \frac{R h_{ij} h_{kl}}{n-2} (g_{ij} R_{kl} + g_{kl} R_{ij} - g_{il} R_{jk} - g_{jk} R_{il}) \\ &\quad + \frac{R^2 h_{ij} h_{kl}}{(n-1)(n-2)} (g_{ij} g_{kl} - g_{il} g_{jk}) \\ &= |h|^2 |Rc|^2 - \frac{2R}{n-2} (H R_{ij} h_{ij} - R_{jk} h_{ji} h_{ik}) + \frac{R^2 (H^2 - |h|^2)}{(n-1)(n-2)} \end{aligned}$$

and where  $H := g^{ij} h_{ij}$ .

Now we need to deal with the troublesome term  $P$ . Fortunately, we claim that  $P$  is always nonnegative without any assumption, which shall be confirmed in the Section 4 (see Corollary 4.4).

**Claim 2.1.** *If  $n \geq 4$ , for any metric  $g$  and symmetric 2-tensor  $h$ , we have*

$$|h|^2 |Rc|^2 - \frac{2R}{n-2} (H R_{ij} h_{ij} - R_{jk} h_{ji} h_{ik}) + \frac{R^2 (H^2 - |h|^2)}{(n-1)(n-2)} \geq 0.$$

Proceeding our proof, by Claim 2.1 and  $R > 0$ , from (6) we derive that

$$\frac{\partial}{\partial t} \left( \frac{|h|^2}{R^2} \right) \leq \Delta \left( \frac{|h|^2}{R^2} \right) + \frac{2}{R} \nabla R \cdot \nabla \left( \frac{|h|^2}{R^2} \right).$$

Finally, applying the parabolic maximum principle to the above equation yields

$$\frac{|h|^2}{R^2}(x, t) \leq c_0$$

on  $M \times [0, T)$ , where  $c_0 := \max_{t=0} |h|^2 / R^2$ .  $\square$

### 3. PROOF OF THEOREM 1.4

In this section, we will prove Theorem 1.4. First, we give a Riemannian curvature estimate under the Ricci flow on a closed manifold, which will be useful in the proof of Theorem 1.4.

**Lemma 3.1.** *Let  $(M^n, g(t))$  be a solution to the Ricci flow on a closed  $n$ -dimensional manifold on a time interval  $[0, T)$  with  $T < \infty$ . Then*

$$(7) \quad |Rm(x, t)| \leq \frac{K}{1 - 8Kt}$$

on  $M \times [0, T')$ , where  $T' := \min\{T, \frac{1}{8K}\}$  and  $K := \max_{t=0} |Rm|$ .

*Proof.* Under the Ricci flow,

$$\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + 16|Rm|^3.$$

By the maximum principle, we have

$$|Rm(g(t))| \leq \frac{K}{1 - 8Kt}$$

on  $M \times [0, T')$ , where  $T' := \min\{T, \frac{1}{8K}\}$  and  $K := \max_{t=0} |Rm|$ .  $\square$

Now we use Lemma 3.1 to finish the proof of Theorem 1.4.

*Proof of Theorem 1.4.* As before, we still have the evolution equation (5). Since  $(M^n, g(t))$ ,  $t \in [0, T)$ ,  $T < \infty$  has a solution of the Ricci flow (1) on a closed  $n$ -dimensional ( $n \geq 4$ ) manifold, by Lemma 3.1, we have

$$(8) \quad |Rm(x, t)| \leq \frac{K}{1 - 8Kt}$$

on  $M \times [0, T')$ , where  $T' := \min\{T, \frac{1}{8K}\}$  and  $K := \max_{t=0} |Rm|$ . Substituting (8) into (5) and noticing  $R > 0$  by our assumption, we have

$$\frac{\partial}{\partial t} \left( \frac{|h|^2}{R^2} \right) \leq \Delta \left( \frac{|h|^2}{R^2} \right) + \frac{2}{R} \nabla R \cdot \nabla \left( \frac{|h|^2}{R^2} \right) + \frac{C(n)K}{1 - 8Kt} \cdot \frac{|h|^2}{R^2}.$$

Applying the parabolic maximum principle to the above evolution equation (for example, see Proposition 4.3 in [10]), we conclude that

$$\frac{|h|^2}{R^2}(t) \leq c_0 \cdot (1 - 8Kt)^{-c}$$

on  $M \times [0, T']$ , where  $c_0 := \max_{t=0} |h|^2/R^2$  and  $c := c(n)$ . This finishes the proof of the theorem.  $\square$

#### 4. NONNEGATIVITY OF A DEGREE 4 HOMOGENEOUS POLYNOMIAL IN $2n$ VARIABLES

In this section, we will prove Claim 2.1 in Section 2. Our proof seems to be different from [1]. Adapting the Anderson-Chow's symbols in [1], since  $h$  is a symmetric tensor, we may assume  $h$  is diagonal. Let  $h_1, h_2, \dots, h_n$  denote the eigenvalues of  $h$  and let  $r_1 = R_{11}, r_2 = R_{22}, \dots, r_n = R_{nn}$  denote the diagonal entries of  $R_{ij}$ . Then

$$\begin{aligned}
 (9) \quad P &= |Rc|^2 |h|^2 - \frac{2R}{n-2} (HR_{ij}h_{ij} - R_{jk}h_{ji}h_{ik}) + \frac{R^2(H^2 - |h|^2)}{(n-1)(n-2)} \\
 &\geq Q := \sum_{i=1}^n r_i^2 \cdot \sum_{i=1}^n h_i^2 \\
 &\quad + \frac{2}{n-2} \sum_{i=1}^n r_i \cdot \left( - \sum_{i=1}^n h_i \sum_{i=1}^n r_i h_i + \sum_{i=1}^n r_i h_i^2 \right) \\
 &\quad + \frac{1}{(n-1)(n-2)} \left( \sum_{i=1}^n r_i \right)^2 \cdot \left[ \left( \sum_{i=1}^n h_i \right)^2 - \sum_{i=1}^n h_i^2 \right],
 \end{aligned}$$

where we used the fact  $|Rc|^2 \geq \sum_{i=1}^n r_i^2$ .

Now we rewrite  $Q$  as a bilinear form in  $\gamma = (r_1, r_2, \dots, r_n)^T$  with coefficients in  $h = (h_1, h_2, \dots, h_n)^T$ :

$$Q = \gamma^T (s_2 I_n + \alpha_0 \beta^T + \beta \alpha_0^T) \gamma,$$

where  $I_n$  is the identity matrix of order  $n$ ,

$$(10) \quad \alpha_0 := (1, 1, \dots, 1)^T \in \mathbb{R}^n, \quad \alpha_k := (h_1^k, h_2^k, \dots, h_n^k)^T \in \mathbb{R}^n$$

and

$$(11) \quad s_0 = n, \quad s_k := \sum_{i=1}^n h_i^k$$

for  $k = 1, 2, 3, 4$ ; and where

$$(12) \quad \beta := \frac{(s_1^2 - s_2)\alpha_0}{2(n-1)(n-2)} - \frac{s_1\alpha_1}{n-2} + \frac{\alpha_2}{n-2}.$$

In the rest of this section, we shall prove  $Q \geq 0$ . To achieve it, we begin with two technical lemmas.

**Lemma 4.1.** *If the matrix  $\Lambda \in \mathbb{R}^{n \times n}$  is nonsingular, then for any column vector  $\xi_i, \eta_i \in \mathbb{R}^n$ , ( $i = 1, 2$ )*

$$\det(\Lambda + \xi_1 \eta_1^T + \xi_2 \eta_2^T) = \det(\Lambda) \cdot \det(I_2 + (\eta_1, \eta_2)^T \Lambda^{-1} (\xi_1, \xi_2)).$$

*Proof.* One can easily check that

$$\begin{bmatrix} I_2 & (\eta_1, \eta_2)^T \\ 0 & \Lambda + \xi_1 \eta_1^T + \xi_2 \eta_2^T \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ (\xi_1, \xi_2) & I_n \end{bmatrix} \times \begin{bmatrix} I_2 & (\eta_1, \eta_2)^T \Lambda^{-1} \\ 0 & I_n \end{bmatrix} \\ \times \begin{bmatrix} I_2 + (\eta_1, \eta_2)^T \Lambda^{-1} (\xi_1, \xi_2) & 0 \\ -(\xi_1, \xi_2) & \Lambda \end{bmatrix}.$$

Then the conclusion can be immediately obtained by the above identity.  $\square$

**Lemma 4.2.** *Let  $\alpha_0$ ,  $\alpha_k$ ,  $s_k$  and  $\beta$  be defined by (10), (11) and (12). Then*

$$(13) \quad s_2 + \alpha_0^T \beta \geq 0$$

and

$$(14) \quad (s_2 + \alpha_0^T \beta)^2 - n\beta^T \beta \geq 0.$$

*Proof.* We first check (13). Clearly,  $\alpha_i^T \alpha_j = s_{i+j}$  for all  $i, j = 0, 1, 2$ . Using this, by the definition  $\beta$ , direct computation yields

$$\begin{aligned} s_2 + \alpha_0^T \beta &= s_2 + \frac{n(s_1^2 - s_2)}{2(n-1)(n-2)} - \frac{s_1^2}{n-2} + \frac{s_2}{n-2} \\ &= \left(1 + \frac{1}{2(n-1)}\right) s_2 - \frac{s_1^2}{2(n-1)} \\ &= \alpha_1^T \left[ \left(1 + \frac{1}{2(n-1)}\right) I_n - \frac{\alpha_0 \alpha_0^T}{2(n-1)} \right] \alpha_1. \end{aligned}$$

The fact that  $1 > \sum_{i=1}^{n-1} \frac{1}{2(n-1)} = \frac{1}{2}$  tells us that the real matrix

$$A := \left(1 + \frac{1}{2(n-1)}\right) I_n - \frac{\alpha_0 \alpha_0^T}{2(n-1)}$$

is strictly diagonally dominant. Moreover all main diagonal entries of  $A$  are positive. Hence, by Theorem 6.1.10 of [17], we conclude that  $A$  is positive definite and  $s_2 + \alpha_0^T \beta = \alpha_1^T A \alpha_1 \geq 0$ .

Then, we shall prove (14). According to the definitions  $\alpha_k$  and  $s_k$ , using the relation  $\alpha_i^T \alpha_j = s_{i+j}$ , we expand the expression  $(s_2 + \alpha_0^T \beta)^2 - n\beta^T \beta$  as

$$\begin{aligned} (s_2 + \alpha_0^T \beta)^2 - n\beta^T \beta &= \left[ s_2 + \frac{n(s_1^2 - s_2)}{2(n-1)(n-2)} - \frac{s_1^2}{n-1} + \frac{s_2}{n-1} \right]^2 \\ &\quad - n \left[ \frac{n(s_1^2 - s_2)^2}{4(n-1)^2(n-2)^2} + \frac{s_1^2 s_2 + s_4 - 2s_1 s_3}{(n-2)^2} + \frac{(s_1^2 - s_2)s_2 - (s_1^2 - s_2)s_1^2}{(n-1)(n-2)^2} \right]. \end{aligned}$$

From above, we easily judge that

$$f(h_1, h_2, \dots, h_n) := (s_2 + \alpha_0^T \beta)^2 - n\beta^T \beta$$

is a real homogeneous symmetric polynomial of degree 4 with respect to  $n$  variables:  $h_1, h_2, \dots, h_n$ . Now we recall an interesting fact due to V. Timofte (see Corollary 5.6 in [21]).

**Theorem C** (Timofte [21]). *If  $f$  is a real homogeneous symmetric polynomial of degree 4 on  $\mathbb{R}^n$ , then*

$$f(h_1, h_2, \dots, h_n) \geq 0 \iff \varphi_k(t) \geq 0, t \in [-1, 1], k = 1, 2, \dots, n-1,$$

where

$$\varphi_k(t) := f(h_1, h_2, \dots, h_n)|_{h_1=h_2=\dots=h_k=t, h_{k+1}=\dots=h_n=1}$$

for  $k = 1, 2, \dots, n-1$ .

In order to prove (14), by Theorem C, we only need to check  $\varphi_k(t) \geq 0$  for all  $k = 1, 2, \dots, n-1$ . Indeed,

$$\varphi_1(t) = (t-1)^2 t^2 \geq 0, \quad \varphi_{n-1}(t) = (t-1)^2 \geq 0$$

and

$$\varphi_k(t) = \frac{k(n-k)(t-1)^2}{(n-1)(n-2)^2} (a_1 t^2 + b_1 t + c_1), \quad k = 2, \dots, n-2$$

where

$$\begin{aligned} a_1 &:= [(n-1)k-1](n-1-k) > 0, \\ b_1 &:= -2(k-1)(n-1-k)(n-1), \\ c_1 &:= (k-1)[(n-k)(n-1)-1]. \end{aligned}$$

Since  $a_1 > 0$  and

$$b_1^2 - 4a_1c_1 = -4(k-1)(n-1-k)n(n-2)^2 < 0,$$

we conclude that

$$a_1 t^2 + b_1 t + c_1 > 0 \quad \text{and} \quad \varphi_k(t) \geq 0$$

for  $k = 2, \dots, n-2$ . Therefore (14) follows.  $\square$

Using Lemmas 4.1 and 4.2, we now finish the proof of the nonnegativity of  $Q$ .

**Theorem 4.3.** *Let  $\alpha_0$ ,  $\alpha_k$ ,  $s_k$  and  $\beta$  be defined by (10), (11) and (12). Then*

$$Q = \gamma^T (s_2 I_n + \alpha_0 \beta^T + \beta \alpha_0^T) \gamma \geq 0.$$

*Proof of Theorem 4.3.* Let  $B := s_2 I_n + \alpha_0 \beta^T + \beta \alpha_0^T$ . We only to show that all eigenvalues of real matrix  $B$  are real, nonnegative, and their number is  $n$ . In fact, if  $\lambda \neq s_2$ , by Lemma 4.1, we compute that

$$\begin{aligned} \det(\lambda I_n - B) &= \det((\lambda - s_2)I_n - \alpha_0 \beta^T - \beta \alpha_0^T) \\ &= \det((\lambda - s_2)I_n) \cdot \det\left(I_2 + \frac{1}{(\lambda - s_2)}(\beta, \alpha_0)^T(-\alpha_0, -\beta)\right) \\ &= (\lambda - s_2)^{n-2} \det((\lambda - s_2)I_2 - (\beta, \alpha_0)^T(\alpha_0, \beta)) \\ &= (\lambda - s_2)^{n-2}(\lambda^2 - p\lambda + q), \end{aligned}$$

where

$$p := 2(s_2 + \alpha_0^T \beta) \quad \text{and} \quad q := (s_2 + \alpha_0^T \beta)^2 - n\beta^T \beta.$$

In other words, when  $\lambda \neq s_2$ , we have the identity

$$(15) \quad \det(\lambda I_n - B) = (\lambda - s_2)^{n-2}(\lambda^2 - p\lambda + q).$$

Notice that two hand sides of the above identity are continuous with respect to the parameter  $\lambda$ . Thus if  $\lambda \rightarrow s_2$ , we know that (15) also holds for  $\lambda = s_2$ . Therefore, (15) in fact holds *without* any condition. Now from (15), we easily conclude that  $s_2$  is an nonnegative eigenvalue of multiplicity  $n-2$  in  $B$ .

On the other hand, by Lemma 4.2, we know that  $p, q \geq 0$ . Meanwhile,

$$p^2 - 4q = n\beta^T \beta \geq 0.$$

Hence, equation  $\lambda^2 - p\lambda + q = 0$  has two real solutions, which implies that  $B$  has another two nonnegative real eigenvalues.

In summary, we prove that the number of nonnegative real eigenvalues of matrix  $B$  is  $n$ . Therefore  $Q = \gamma^T B \gamma \geq 0$ .  $\square$

Combining Theorem 4.3 and (9) implies that

**Corollary 4.4.** *If  $n \geq 4$ , for any Riemannian metric  $g$  and symmetric 2-tensor  $h$ , we have*

$$|h|^2 |Rc|^2 - \frac{2R}{n-2} (H R_{ij} h_{ij} - R_{jk} h_{ji} h_{ik}) + \frac{R^2 (H^2 - |h|^2)}{(n-1)(n-2)} \geq 0.$$

**Remark 4.5.** *We would like to point out that our proof method here is also suitable for the case  $n = 3$ , which has been proved by G. Anderson and B. Chow [1].*

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